

ON ROW SEQUENCES OF PADÉ AND HERMITE-PADÉ APPROXIMATION

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Dedicated to my friend Ed on the occasion of his 70-th birthday

ABSTRACT. A survey of direct and inverse type results for row sequences of Padé and Hermite-Padé approximation is given. A conjecture is posed on an inverse type result for type II Hermite-Padé approximation when it is known that the sequence of common denominators of the approximating vector rational functions has a limit. Some inverse type results are proved for the so called incomplete Padé approximants which may lead to the proof of the conjecture and the connection is discussed.

Keywords Montessus de Ballore Theorem · Simultaneous approximation · Hermite-Padé approximation · Rate of convergence · Inverse results

Mathematics Subject Classification (2010) Primary 30E10 · 41A21 · 41A28 · Secondary 41A25 · 41A27

1. INTRODUCTION

The study of direct and inverse type results for sequences of rational functions with a fixed number of free poles has been a subject of constant interest in the research of E.B. Saff. In different contexts (multi-point Padé [20], best rational [18]–[19], Hermite-Padé [12]–[14], and Padé orthogonal approximations [2]–[4]) such results are related with Montessus de Ballore's classical theorem [7] on the convergence of the m -th row of the Padé table associated with a formal Taylor expansion

$$(1) \quad f(z) = \sum_{n \geq 0} \phi_n z^n$$

provided that it represents a meromorphic function with exactly m poles (counting multiplicities) in an open disk centered at the origin, and its converse due to A.A. Gonchar [10, Section 3, Subsection 4], [11, Section 2] which allows to deduce analytic properties of f if it is known that the poles of the approximants converge with geometric rate.

Let $m \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ be fixed. If f is analytic at the origin, $D_m(f)$ denotes the largest open disk centered at the origin to which f may be extended as a meromorphic function with at most m poles and $R_m(f)$ is its radius; otherwise, we take $D_m(f) = \emptyset$ and $R_m(f) = 0$ for each $m \in \mathbb{Z}_+$.

Date: February 16, 2015.

The value $R_m(f)$ may be calculated, as shown by J. Hadamard [15], in terms of the Taylor coefficients ϕ_n . Let $\mathcal{P}_m(f)$ be the set of poles in $D_m(f)$. By $(R_{n,m})_{n \geq 0}, m \in \mathbb{Z}_+$ fixed, we denote the m -th row of the Padé table associated with f , see Definition 1.2 restricted to $d = 1$.

The combined Montessus de Ballore–Gonchar theorem may be formulated in the following terms

Theorem 1.1. *Let f be a formal Taylor expansion about the origin and fix $m \in \mathbb{N} = \{1, 2, \dots\}$. Then, the following two assertions are equivalent.*

- a) $R_0(f) > 0$ and f has exactly m poles in $D_m(f)$ counting multiplicities.
- b) There is a monic polynomial Q_m of degree m , $Q_m(0) \neq 0$, such that the sequence of denominators $(Q_{n,m})_{n \geq 0}$ of the Padé approximations of f , taken with leading coefficient equal to 1, satisfies

$$\limsup_{n \rightarrow \infty} \|Q_m - Q_{n,m}\|^{1/n} = \theta < 1,$$

where $\|\cdot\|$ denotes the ℓ^1 coefficient norm in the space of polynomials.

Moreover, if either a) or b) takes place, the zeros of Q_m are the poles of f in $D_m(f)$,

$$(2) \quad \theta = \frac{\max\{|\xi| : \xi \in \mathcal{P}_m(f)\}}{R_m(f)},$$

and

$$(3) \quad \limsup_{n \rightarrow \infty} \|f - R_{n,m}\|_K^{1/n} = \frac{\|z\|_K}{R_m(f)},$$

where K is any compact subset of $D_m(f) \setminus \mathcal{P}_m(f)$.

Since all norms in finite dimensional spaces are equivalent in b) any other norm in the $m + 1$ dimensional space of polynomials of degree $\leq m$ would do as well.

From Theorem 1.1 it follows that if ξ is a pole of f in $D_m(f)$ of order τ , then for each $\varepsilon > 0$, there exists n_0 such that for $n \geq n_0$, $Q_{n,m}$ has exactly τ zeros in $\{z : |z - \xi| < \varepsilon\}$. We say that each pole of f in $D_m(f)$ attracts as many zeros of $Q_{n,m}$ as its order when n tends to infinity.

Under assumptions a), in [7] Montessus de Ballore proved that

$$\lim_{n \rightarrow \infty} Q_{n,m} = Q_m, \quad \lim_{n \rightarrow \infty} R_{n,m} = f,$$

with uniform convergence on compact subsets of $D_m(f) \setminus \mathcal{P}_m(f)$ in the second limit. In essence, Montessus proved that a) implies b), showed that $\theta \leq \max\{|\xi| : \xi \in \mathcal{P}_m(f)\}/R_m(f)$, and proved (3) with equality replaced by \leq . These are the so called direct statements of the theorem. The inverse statements, b) implies a), $\theta \geq \max\{|\xi| : \xi \in \mathcal{P}_m(f)\}/R_m(f)$, and the inequality \geq in (3) are immediate consequences of [10, Theorem 1]. The study of inverse problems when the behavior of individual sequences of poles of the approximants is known was suggested by A.A. Gonchar in [10, Subsection

[12] where he presented some interesting conjectures. Some of them were solved in [21] and [22] by S.P. Suetin.

In [12], Graves-Morris and Saff proved an analogue of Montessus' theorem for Hermite-Padé (vector rational) approximation with the aid of the concept of polewise independence of a system of functions.

Let $\mathbf{f} = (f_1, \dots, f_d)$ be a system of d formal or convergent Taylor expansions about the origin; that is, for each $k = 1, \dots, d$, we have

$$(4) \quad f_k(z) = \sum_{n=0}^{\infty} \phi_{n,k} z^n, \quad \phi_{n,k} \in \mathbb{C}.$$

Let $\mathbf{D} = (D_1, \dots, D_d)$ be a system of domains such that, for each $k = 1, \dots, d$, f_k is meromorphic in D_k . We say that the point ξ is a pole of \mathbf{f} in \mathbf{D} of order τ if there exists an index $k \in \{1, \dots, d\}$ such that $\xi \in D_k$ and it is a pole of f_k of order τ , and for $j \neq k$ either ξ is a pole of f_j of order less than or equal to τ or $\xi \notin D_j$. When $\mathbf{D} = (D, \dots, D)$ we say that ξ is a pole of \mathbf{f} in D .

Let $R_0(\mathbf{f})$ be radius of the largest open disk $D_0(\mathbf{f})$ in which all the expansions $f_k, k = 1, \dots, d$ correspond to analytic functions. If $R_0(\mathbf{f}) = 0$, we take $D_m(\mathbf{f}) = \emptyset, m \in \mathbb{Z}_+$; otherwise, $R_m(\mathbf{f})$ is the radius of the largest open disk $D_m(\mathbf{f})$ centered at the origin to which all the analytic elements $(f_k, D_0(f_k))$ can be extended so that \mathbf{f} has at most m poles counting multiplicities. The disk $D_m(\mathbf{f})$ constitutes for systems of functions the analogue of the m -th disk of meromorphy defined by J. Hadamard in [15] for $d = 1$. Moreover, in that case both definitions coincide.

By $\mathcal{Q}_m(\mathbf{f})$ we denote the monic polynomial whose zeros are the poles of \mathbf{f} in $D_m(\mathbf{f})$ counting multiplicities. The set of distinct zeros of $\mathcal{Q}_m(\mathbf{f})$ is denoted by $\mathcal{P}_m(\mathbf{f})$.

Definition 1.2. Let $\mathbf{f} = (f_1, \dots, f_d)$ be a system of d formal Taylor expansions as in (4). Fix a multi-index $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ where $\mathbf{0}$ denotes the zero vector in \mathbb{Z}_+^d . Set $|\mathbf{m}| = m_1 + \dots + m_d$. Then, for each $n \geq \max\{m_1, \dots, m_d\}$, there exist polynomials $Q, P_k, k = 1, \dots, d$, such that

- a.1) $\deg P_k \leq n - m_k, k = 1, \dots, d, \quad \deg Q \leq |\mathbf{m}|, \quad Q \not\equiv 0,$
- a.2) $Q(z)f_k(z) - P_k(z) = A_k z^{n+1} + \dots$

The vector rational function $\mathbf{R}_{n,\mathbf{m}} = (P_1/Q, \dots, P_d/Q)$ is called an (n, \mathbf{m}) (type II) Hermite-Padé approximation of \mathbf{f} .

Type I and type II Hermite-Padé approximation were introduced by Ch. Hermite and used in the proof of the transcendence of e , see [16]. We will only consider here type II and, for brevity, will be called Hermite-Padé approximants.

In contrast with Padé approximation, such vector rational approximants, in general, are not uniquely determined and in the sequel we assume that given (n, \mathbf{m}) one particular solution is taken. For that solution we write

$$(5) \quad \mathbf{R}_{n,\mathbf{m}} = (R_{n,\mathbf{m},1}, \dots, R_{n,\mathbf{m},d}) = (P_{n,\mathbf{m},1}, \dots, P_{n,\mathbf{m},d})/Q_{n,\mathbf{m}},$$

where $Q_{n,\mathbf{m}}$ is a monic polynomial that has no common zero simultaneously with all the $P_{n,\mathbf{m},k}$. Sequences $(\mathbf{R}_{n,\mathbf{m}})_{n \geq |\mathbf{m}|}$, for which \mathbf{m} remains fixed when n varies are called row sequences.

For each $r > 0$, set $D_r = \{z \in \mathbb{C} : |z| < r\}$, $\Gamma_r = \{z \in \mathbb{C} : |z| = r\}$, and $\overline{D}_r = \{z \in \mathbb{C} : |z| \leq r\}$.

Definition 1.3. Let $\mathbf{f} = (f_1, \dots, f_d)$ be a system of meromorphic functions in the disk D_r and let $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$. We say that the system \mathbf{f} is polewise independent with respect to \mathbf{m} in D_r if there do not exist polynomials p_1, \dots, p_d , at least one of which is non-null, such that

- b.1) $\deg p_k < m_k$ if $m_k \geq 1$, $k = 1, \dots, d$,
- b.2) $p_k \equiv 0$ if $m_k = 0$, $k = 1, \dots, d$,
- b.3) $\sum_{k=1}^d p_k f_k$ is analytic on D_r .

In [12, Theorem 1], Graves-Morris and Saff established an analogue of the direct part of the previous theorem when \mathbf{f} is polewise independent with respect to \mathbf{m} in $D_{|\mathbf{m}|}(\mathbf{f})$ obtaining upper bounds for the convergence rates corresponding to (2) and (3). It should be stressed that [12] was pioneering in the sense that it initiated a convergence theory for Hermite-Padé approximation.

The result [12, Theorem 1] does not allow a converse statement in the sense of Gonchar's theorem. Inspired in the concept of polewise independence, in [6] we proposed the following relaxed version of it.

Definition 1.4. Given $\mathbf{f} = (f_1, \dots, f_d)$ and $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ we say that $\xi \in \mathbb{C} \setminus \{0\}$ is a system pole of order τ of \mathbf{f} with respect to \mathbf{m} if τ is the largest positive integer such that for each $s = 1, \dots, \tau$ there exists at least one polynomial combination of the form

$$(6) \quad \sum_{k=1}^d p_k f_k, \quad \deg p_k < m_k, \quad k = 1, \dots, d,$$

which is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole at $z = \xi$ of exact order s . If some component m_k equals zero the corresponding polynomial p_k is taken identically equal to zero.

The advantage of this definition with respect to that of polewise independence is that it does not require to determine a priori a region where the property should be verified. Polewise independence of \mathbf{f} in $D_{|\mathbf{m}|}(\mathbf{f})$ with respect to \mathbf{m} implies that \mathbf{f} has in $D_{|\mathbf{m}|}$ exactly $|\mathbf{m}|$ system poles (counting their order).

We wish to underline that if some component m_k equals zero, that component places no restriction on Definition 1.2 and does not report any benefit in finding system poles; therefore, without loss of generality one can restrict the attention to multi-indices $\mathbf{m} \in \mathbb{N}^d$.

A system \mathbf{f} cannot have more than $|\mathbf{m}|$ system poles with respect to \mathbf{m} counting their order. A system pole need not be a pole of \mathbf{f} and a pole may not be a system pole, see examples in [6].

To each system pole ξ of \mathbf{f} with respect to \mathbf{m} one can associate several characteristic values. Let τ be the order of ξ as a system pole of \mathbf{f} . For each $s = 1, \dots, \tau$ denote by $r_{\xi,s}(\mathbf{f}, \mathbf{m})$ the largest of all the numbers $R_s(g)$ (the radius of the largest disk containing at most s poles of g), where g is a polynomial combination of type (6) that is analytic on a neighborhood of $\overline{D}_{|\xi|}$ except for a pole at $z = \xi$ of order s . Set

$$R_{\xi,s}(\mathbf{f}, \mathbf{m}) := \min_{k=1, \dots, s} r_{\xi,k}(\mathbf{f}, \mathbf{m}),$$

$$R_{\xi}(\mathbf{f}, \mathbf{m}) := R_{\xi,\tau}(\mathbf{f}, \mathbf{m}) = \min_{s=1, \dots, \tau} r_{\xi,s}(\mathbf{f}, \mathbf{m}).$$

Obviously, if $d = 1$ and $(\mathbf{f}, \mathbf{m}) = (f, m)$, system poles and poles in $D_m(f)$ coincide. Also, $R_{\xi}(\mathbf{f}, \mathbf{m}) = R_m(f)$ for each pole ξ of f in $D_m(f)$.

Let $\mathcal{Q}(\mathbf{f}, \mathbf{m})$ denote the monic polynomial whose zeros are the system poles of \mathbf{f} with respect to \mathbf{m} taking account of their order. The set of distinct zeros of $\mathcal{Q}(\mathbf{f}, \mathbf{m})$ is denoted by $\mathcal{P}(\mathbf{f}, \mathbf{m})$. We have (see [6, Theorem 1.4])

Theorem 1.5. *Let \mathbf{f} be a system of formal Taylor expansions as in (4) and fix a multi-index $\mathbf{m} \in \mathbb{N}^d$. Then, the following assertions are equivalent.*

- a) $R_0(\mathbf{f}) > 0$ and \mathbf{f} has exactly $|\mathbf{m}|$ system poles with respect to \mathbf{m} counting multiplicities.
- b) The denominators $Q_{n,\mathbf{m}}$, $n \geq |\mathbf{m}|$, of simultaneous Padé approximations of \mathbf{f} are uniquely determined for all sufficiently large n and there exists a polynomial $Q_{|\mathbf{m}|}$ of degree $|\mathbf{m}|$, $Q_{|\mathbf{m}|}(0) \neq 0$, such that

$$\limsup_{n \rightarrow \infty} \|Q_{|\mathbf{m}|} - Q_{n,\mathbf{m}}\|^{1/n} = \theta < 1.$$

Moreover, if either a) or b) takes place then $Q_{|\mathbf{m}|} \equiv \mathcal{Q}(\mathbf{f}, \mathbf{m})$ and

$$(7) \quad \theta = \max \left\{ \frac{|\xi|}{R_{\xi}(\mathbf{f}, \mathbf{m})} : \xi \in \mathcal{P}(\mathbf{f}, \mathbf{m}) \right\}.$$

If $d = 1$, $R_{n,m}$ and $Q_{n,m}$ are uniquely determined; therefore, Theorem 1.5 contains Theorem 1.1. The analogue of (3) is found in [6, Theorem 3.7]).

In the rest of the paper we wish to discuss the case when

$$(8) \quad \lim_{n \rightarrow \infty} Q_{n,\mathbf{m}} = Q_{|\mathbf{m}|}, \quad \deg Q_{|\mathbf{m}|} = |\mathbf{m}|, \quad Q_{|\mathbf{m}|}(0) \neq 0,$$

but the rate of convergence is not known in advance. Now the reference in the scalar case is a result by S.P. Suetin [22].

Theorem 1.6. *Assume that $\lim_{n \rightarrow \infty} Q_{n,m}(z) = Q_m(z) = \prod_{k=1}^m (z - z_k)$ and $0 < |z_1| \leq \dots \leq |z_N| < |z_{N+1}| = \dots = |z_m| = R$.*

Then z_1, \dots, z_N are the poles of f in $D_{m-1}(f)$ (taking account of their order), $R_N(f) = \dots = R_{m-1}(f) = R$, and z_{N+1}, \dots, z_m are singularities of f on the boundary of $D_{m-1}(f)$.

When $m = 1$ it is easy to see from the definition that $Q_{n,1} = z - (\phi_n/\phi_{n+1})$ whenever $\phi_{n+1} \neq 0$. Therefore, Suetin's theorem contains the classical theorem of E. Fabry [8] which states that $\lim_{n \rightarrow \infty} \phi_n/\phi_{n+1} = \zeta \neq 0$ implies that $R_0(f) = |\zeta|$ and ζ is a singular point of f .

Let us introduce the concept of system singularity of \mathbf{f} with respect to \mathbf{m} .

Definition 1.7. Given $\mathbf{f} = (f_1, \dots, f_d)$ and $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{Z}_+^d \setminus \{\mathbf{0}\}$ we say that $\xi \in \mathbb{C} \setminus \{0\}$ is a system singularity of \mathbf{f} with respect to \mathbf{m} if there exists at least one polynomial combination F of the form (6) analytic on $D_{|\xi|}$ and ξ is a singular point of F .

We believe that the following result holds.

Conjecture. Assume that $Q_{n,\mathbf{m}}$ is unique for all sufficiently large n , (8) takes place, and let $Q_{|\mathbf{m}|}(\zeta) = 0$. Then, ζ is a system singularity of \mathbf{f} with respect to \mathbf{m} . If $\zeta \in D_1(F)$, for some polynomial combination F determines the system singularity of \mathbf{f} at ζ , then ζ is a system pole of \mathbf{f} with respect to \mathbf{m} of order equal to the multiplicity of ζ as a zero of $Q_{|\mathbf{m}|}$.

This conjecture applied to the scalar case reduces to Theorem 1.6.

In Section 2, we give a result similar to Theorem 1.6 for so called incomplete Padé approximation. Such approximants were introduced in [5] and used in [6] to prove Theorem 1.5. In the final section we describe some steps which may lead to the proof of the conjecture.

2. INCOMPLETE PADÉ APPROXIMANTS

Consider the following construction.

Definition 2.1. Let f denote a formal Taylor expansion as in (1). Fix $m \geq m^* \geq 1$. Let $n \geq m$. We say that the rational function $r_{n,m}$ is an incomplete Padé approximation of type (n, m, m^*) corresponding to f if $r_{n,m}$ is the quotient of any two polynomials p and q that verify

- c.1) $\deg p \leq n - m^*$, $\deg q \leq m$, $q \not\equiv 0$,
- c.2) $q(z)f(z) - p(z) = Az^{n+1} + \dots$.

Given (n, m, m^*) , $n \geq m \geq m^*$, the Padé approximants $R_{n,m^*}, \dots, R_{n,m}$ can all be regarded as incomplete Padé approximation of type (n, m, m^*) of f . From Definition 1.2 and (5) it follows that $R_{n,\mathbf{m},k}$, $k = 1, \dots, d$, is an incomplete Padé approximation of type $(n, |\mathbf{m}|, m_k)$ with respect to f_k .

In the sequel, for each $n \geq m \geq m^*$, we choose one incomplete Padé approximant. After canceling out common factors between q and p , we write $r_{n,m} = p_{n,m}/q_{n,m}$, where, additionally, $q_{n,m}$ is normalized as follows

$$(9) \quad q_{n,m}(z) = \prod_{|\zeta_{n,k}| \leq 1} (z - \zeta_{n,k}) \prod_{|\zeta_{n,k}| > 1} \left(1 - \frac{z}{\zeta_{n,k}}\right).$$

Suppose that q and p have a common zero at $z = 0$ of order λ_n . Notice that $0 \leq \lambda_n \leq m$. From c.1)-c.2) it follows that

- c.3) $\deg p_{n,m} \leq n - m^* - \lambda_n, \quad \deg q_{n,m} \leq m - \lambda_n, \quad q_{n,m} \not\equiv 0,$
c.4) $q_{n,m}(z)f(z) - p_{n,m}(z) = Az^{n+1-\lambda_n} + \dots.$

where A is, in general, a different constant from the one in c.2).

From Definition 2.1 it readily follows that for each $n \geq m$

$$(10) \quad r_{n+1,m}(z) - r_{n,m}(z) = \frac{A_{n,m}z^{n+1-\lambda_n-\lambda_{n+1}}q_{n,m-m^*}^*(z)}{q_{n,m}(z)q_{n+1,m}(z)},$$

where $A_{n,m}$ is some constant and $q_{n,m-m^*}^*$ is a polynomial of degree less than or equal to $m - m^*$ normalized as in (9).

The first difficulty encountered in dealing with inverse-type results is to justify in terms of the data that the formal series corresponds to an analytic element around the origin which does not reduce to a polynomial. Set

$$R_m^*(f) = \left(\limsup_{n \rightarrow \infty} |A_{n,m}|^{1/n} \right)^{-1}, \quad D_m^*(f) = \{z : |z| < R_m^*(f)\}.$$

Let B be a subset of the complex plane \mathbb{C} . By $\mathcal{U}(B)$ we denote the class of all coverings of B by at most a numerable set of disks. Set

$$\sigma(B) = \inf \left\{ \sum_{i=1}^{\infty} |U_i| : \{U_i\} \in \mathcal{U}(B) \right\},$$

where $|U_i|$ stands for the radius of the disk U_i . The quantity $\sigma(B)$ is called the 1-dimensional Hausdorff content of the set B . In the papers we refer to below, the only properties used of the 1-dimensional Hausdorff content follow easily from the definition. They are: subadditivity, monotonicity, and that the 1-dimensional Hausdorff content of a disk of radius R and a segment of length d are R and $d/2$, respectively.

Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence of functions defined on a domain $D \subset \mathbb{C}$ and φ another function defined on D . We say that $(\varphi_n)_{n \in \mathbb{N}}$ converges in σ -content to the function φ on compact subsets of D if for each compact subset K of D and for each $\varepsilon > 0$, we have

$$\lim_{n \rightarrow \infty} \sigma\{z \in K : |\varphi_n(z) - \varphi(z)| > \varepsilon\} = 0.$$

We denote this writing $\sigma\text{-}\lim_{n \rightarrow \infty} \varphi_n = \varphi$ inside D .

Using telescopic sums, it is not difficult to prove the following (see [5, Theorem 3.4]).

Lemma 2.2. *Let f be a formal power series as in (1). Fix m and m^* nonnegative integers, $m \geq m^*$. Let $(r_{n,m})_{n \geq m}$ be a sequence of incomplete Padé approximants of type (n, m, m^*) for f . If $R_m^*(f) > 0$ then $R_0(f) > 0$. Moreover,*

$$D_{m^*}(f) \subset D_m^*(f) \subset D_m(f)$$

and $D_m^*(f)$ is the largest disk in compact subsets of which $\sigma\text{-}\lim_{n \rightarrow \infty} r_{n,m} = f$. Moreover, the sequence $(r_{n,m})_{n \geq m}$ is pointwise divergent in $\{z : |z| > R_m^*(f)\}$ except on a set of σ -content zero.

We also have (see [6, Corollaries 2.4, 2.5])

Lemma 2.3. *Let f be a formal power series as in (1). Fix $m \geq m^* \geq 1$. Assume that there exists a polynomial q_m of degree greater than or equal to $m - m^* + 1$, $q_m(0) \neq 0$, such that $\lim_{n \rightarrow \infty} q_{n,m} = q_m$. Then $0 < R_0(f) < \infty$ and the zeros of q_m contain all the poles, counting multiplicities, that f has in $D_m^*(f)$.*

Suppose that $\limsup_n |A_{n,m}|^{1/n} = 1$. It is known, that there exists a regularizing sequence $(A_{n,m}^*)_{n \geq m}$ such that:

- i) $\lim_{n \rightarrow \infty} A_{n,m}^*/A_{n+1,m}^* = 1$,
- ii) $\{\log(A_{n,m}^*/n!)\}$ is concave,
- iii) $|A_{n,m}| \leq |A_{n,m}^*|, n \in \mathbb{Z}_+$,
- iv) $|A_{n,m}| \geq c|A_{n,m}^*|, n \in \Lambda \subset \mathbb{Z}_+, c > 0$ for an infinite sequence Λ .

The use of such regularizing sequences is well established in the study of singularities of Taylor series (see, for example, [1] and [17]). Its use was extended by S.P. Suetin in [22] to Padé approximation for proving Theorem 1.6. The proofs of [22, Lemmas 1, 2] (see also [23, Chapter 1]) may be easily adjusted to produce the following result for incomplete Padé approximation.

Lemma 2.4. *Let f be a formal power series as in (1). Fix $m \geq m^* \geq 1$. Assume that $\lim_{n \rightarrow \infty} |A_{n,m}|^{1/n} = 1$. For any $\delta > 0$*

$$(11) \quad \max_{|z| \geq e^\delta} |p_{n,m}(z)/(A_{n,m}^* z^n)| = \mathcal{O}(1), \quad n \rightarrow \infty.$$

For every compact $K \subset \{z : |z| < e^{-\delta}\} \setminus \mathcal{P}(f)$,

$$(12) \quad \max_K |(q_{n,m} f - p_{n,m})(z)/(A_{n,m}^* z^n)| = \mathcal{O}(1), \quad n \rightarrow \infty.$$

Assume that there exists a polynomial q_m , $\deg q_m = m$, $q_m(0) \neq 0$, such that

$$\lim_{n \rightarrow \infty} q_{n,m} = q_m.$$

Let f be holomorphic in some region $G \supset D_m^*(f) \setminus \mathcal{P}(f)$. Then, for every compact $K \subset G$, (12) takes place.

In the sequel $\text{dist}(\zeta, B_n)$ denotes the distance from a point ζ to a set B_n . Let $\mathcal{P}_{n,m}(f) = \{\zeta_{n,1}, \dots, \zeta_{n,m}\}$ be the set of zeros of $q_{n,m}$ and the points are enumerated so that

$$|\zeta_{n,1} - \zeta| \leq \dots \leq |\zeta_{n,m_n} - \zeta|.$$

We say that $\lambda = \lambda(\zeta)$ points of $\mathcal{P}_{n,m}$ tend to ζ if

$$\lim_{n \rightarrow \infty} |\zeta_{n,\lambda} - \zeta| = 0, \quad \limsup_{n \rightarrow \infty} |\zeta_{n,\lambda+1} - \zeta| > 0.$$

By convention, $\limsup_{n \rightarrow \infty} |\zeta_{n,\kappa} - \zeta| > 0$ for $\kappa > \liminf_{n \rightarrow \infty} m_n$.

Theorem 2.5. *Let f be a formal power series as in (1). Fix $m \geq m^* \geq 1$. Assume that $0 < R_m^*(f) < +\infty$. Suppose that*

$$\lim_{n \rightarrow \infty} \text{dist}(\zeta, \mathcal{P}_{n,m}(f)) = 0.$$

Let $\mathcal{Z}_n(f)$ be the set of zeros of $q_{n,m-m^*}^*(f)$. If $|\zeta| > R_m^*(f)$, then

$$(13) \quad \lim_{n \in \Lambda} \text{dist}(\zeta, \mathcal{Z}_n(f)) = 0$$

where Λ is any infinite sequence of indices verifying iv) in the regularization of $(A_{n,m})_{n \geq m}$. If $|\zeta| < R_m^*(f)$, then either (13) takes place or ζ is a pole of f of order greater or equal to $\lambda(\zeta)$. If $\lim_{n \rightarrow \infty} q_{n,m} = q_m$, $\deg q_m = m$, $q_m(0) \neq 0$, and $|\zeta| = R_m^*(f)$ then we have either (13) or ζ is a singular point. If the zeros of q_m are distinct then at least m^* of them are singular points of f and lie in the closure of $D_m^*(f)$, those lying in $D_m^*(f)$ are simple poles.

Proof. Without loss of generality, we can assume that $R_m^*(f) = 1$. The general case reduces to it with the change of variables $z \rightarrow z/R_m^*(f)$. Assume that $|\zeta| \neq 1$ and ζ is a regular point of f should $|\zeta| < 1$. Choose $\delta > 0$ such that $|\zeta| > e^\delta$ or $|\zeta| < e^{-\delta}$ depending on whether $|\zeta| > 1$ or $|\zeta| < 1$, respectively. Let $q_{n,m}(\zeta_n) = 0$, $\lim_{n \rightarrow \infty} \zeta_n = \zeta$.

Evaluating at ζ_n , using (11), if $|\zeta| > 1$ or (12), when $|\zeta| < 1$, and taking iv) into account, it follows that

$$|p_{n,m}(\zeta_n)/(A_{n,m}\zeta_n^n)| \leq C_1, \quad n \geq n_0, \quad n \in \Lambda,$$

where C_1 is some constant and Λ is the sequence of indices which appears in the regularization of $(A_{n,m})_{n \geq m}$. (In the sequel C_1, C_2, \dots denote constants which do not depend on n .) However, from (10) it follows that

$$p_{n,m}(\zeta_n)/(A_{n,m}\zeta_n^n) = -\zeta_n^{1-\lambda_n-\lambda_{n+1}} q_{n,m-m^*}^*(\zeta_n)/q_{n+1,m}(\zeta_n),$$

which combined with the previous inequality gives

$$|q_{n,m-m^*}^*(\zeta_n)| \leq C_2 |q_{n+1,m}(\zeta_n)|, \quad n \geq n_0, \quad n \in \Lambda.$$

Therefore, (13) takes place.

If $|\zeta| = 1$ and ζ is a regular point the proof of (13) is the same as for the case when $|\zeta| < 1$. In this case use (12) on a closed neighborhood of ζ contained in $G \supset D_m^*(f) \setminus \mathcal{P}(f)$.

Now, assume that $|\zeta| < 1$ and $\limsup_{n \in \Lambda} \text{dist}(\zeta, \mathcal{Z}_n(f)) > 0$. Then, ζ is a singular point of f . Since $D_m^*(f) \subset D_m(f)$ according to Lemma 2.2, ζ must be a pole of f . Let τ be the order of the pole of f at ζ . Set $w(z) = (z - \zeta)^\tau$ and $F = wf$. Notice that $F(\zeta) \neq 0$. Using (12) and iv), it follows that there exists a closed disk U_r centered at ζ of radius r sufficiently small so that

$$(14) \quad \max_{U_r} |(q_{n,m}F - p_{n,m}w)(z)/(A_{n,m}z^n)| \leq C_3, \quad n \geq n_0, \quad n \in \Lambda.$$

Suppose that $\tau < \lambda(\zeta)$. Since $\sigma - \lim_{n \rightarrow \infty} r_{n,m} = f$ (see Lemma 2.2), it follows that for each $n \in \mathbb{Z}_+$ there exists a zero of η_n of $p_{n,m}$ such that $\lim_{n \rightarrow \infty} \eta_n = \zeta$. Take $r > 0$ sufficiently small so that $\min_{U_r} |F(z)| > 0$. Substituting η_n in (14), we have

$$|q_{n,m}(\eta_n)/(A_{n,m}\eta_n^n)| \leq C_4, \quad n \geq n_0, \quad n \in \Lambda,$$

and taking into account that (10) leads to

$$q_{n,m}(\eta_n)/(A_{n,m}\eta_n^n) = \eta_n^{1-\lambda_n-\lambda_{n+1}} q_{n,m-m^*}^*(\eta_n)/p_{n+1,m}(\eta_n),$$

we obtain

$$|q_{n,m-m^*}^*(\eta_n)| \leq C_5 |p_{n+1,m}(\eta_n)|, \quad n \geq n_0, \quad n \in \Lambda.$$

Since $\limsup_{n \in \Lambda} \text{dist}(\zeta, \mathcal{Z}_n(f)) > 0$, it follows that

$$(15) \quad \lim_{n \in \Lambda'} |p_{n+1,m}(\eta_n)| > 0.$$

for some subsequence $\Lambda' \subset \Lambda$.

The normalization (9) imposed on $q_{n,m}$ implies that for any compact $K \subset \mathbb{C}$ we have $\sup_n \max_K |q_{n,m}(z)| \leq C_6$. So, any sequence $(q_{n,m})_{n \in I}, I \subset \mathbb{Z}_+$, contains a subsequence which converges uniformly on any compact subset of \mathbb{C} . This, combined with $\sigma\text{-}\lim_{n \rightarrow \infty} r_{n,m} = f$ in $D_m^*(f)$, and the assumption that $\tau < \lambda(\zeta)$ imply that there exists a sequence of indices $\Lambda'' \subset \Lambda'$ such that $\lim_{n \in \Lambda''} p_{n+1,m} = F_1$ uniformly on a closed neighborhood of ζ , where F_1 is analytic at ζ and $F_1(\zeta) = 0$ (see [9, Lemma 1] where it is shown that under adequate assumptions uniform convergence on compact subsets of a region can be derived from convergence in 1-dimensional Hausdorff content). This contradicts (15). Thus, $\tau \geq \lambda(\zeta)$ as claimed.

To complete the proof recall that $\deg q_{n,m-m^*}^* \leq m - m^*$ for all $n \geq m$. In particular, this implies that for each $n \in \Lambda$ the set $\mathcal{Z}_n(f)$ has at most $m - m^*$ points. Each zeros ζ of q_m such that either $|\zeta| > 1$ or $|\zeta| \leq 1$ and is regular attracts a sequence of points in $\mathcal{Z}_n(f), n \in \Lambda$. This is clearly impossible if the total number M of such zeros of q_m exceeds $m - m^*$. So, $M \leq m - m^*$. The complement is made up of zeros of q_m which are singular and lie in the closure of $D_m^*(f)$. Those lying in $D_m^*(f)$ are simple poles according to Lemma 2.3. \square

3. SIMULTANEOUS APPROXIMATION

Throughout this section, $\mathbf{f} = (f_1, \dots, f_d)$ denotes a system of formal power expansions as in (4) and $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ is a fixed multi-index. We are concerned with the simultaneous approximation of \mathbf{f} by sequences of vector rational functions defined according to Definition 1.2 taking account of (5). That is, for each $n \in \mathbb{N}, n \geq |\mathbf{m}|$, let $(R_{n,\mathbf{m},1}, \dots, R_{n,\mathbf{m},d})$ be a Hermite-Padé approximation of type (n, \mathbf{m}) corresponding to \mathbf{f} .

As we mentioned earlier, $R_{n,\mathbf{m},k}$ is an incomplete Padé approximant of type $(n, |\mathbf{m}|, m_k)$ with respect to $f_k, k = 1, \dots, d$. Thus, from Lemma 2.2

$$D_{m_k}(f_k) \subset D_{|\mathbf{m}|}^*(f_k) \subset D_{|\mathbf{m}|}(f_k), \quad k = 1, \dots, d.$$

Definition 3.1. A vector $\mathbf{f} = (f_1, \dots, f_d)$ of formal power expansions is said to be polynomially independent with respect to $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ if there do not exist polynomials p_1, \dots, p_d , at least one of which is non-null, such that

- d.1) $\deg p_k < m_k, k = 1, \dots, d,$
- d.2) $\sum_{k=1}^d p_k f_k$ is a polynomial.

In particular, polynomial independence implies that for each $k = 1, \dots, d$, f_k is not a rational function with at most $m_k - 1$ poles. Notice that polynomial independence may be verified solely in terms of the coefficients of the formal Taylor expansions defining the system \mathbf{f} . The system \mathbf{f} is polynomially independent with respect to \mathbf{m} if for all $n \geq n_0$ the polynomial $Q_{n,\mathbf{m}}$ is unique and $\deg Q_{n,\mathbf{m}} = |\mathbf{m}|$, see [6, Lemma 3.2].

An approach to the proof of the conjecture could be

- Remove the restriction in the last part of Theorem 2.5 that the zeros of q_m are distinct.
- Assuming (8), apply the improved version of Theorem 2.5 to the components of \mathbf{f} .
- Using polynomial combinations of the form (6) prove that each zero of $Q_{|\mathbf{m}|}$ is a system singularity. It is sufficient to consider multi-indices of the form $\mathbf{m} = (1, 1, \dots, 1)$ (see beginning of [6, Section 3] for the justification); then, (6) reduces to linear combinations.
- Prove the last part of the conjecture using the final statement of Lemma 2.3.

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